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A unitary similarity transform of a normal matrix to complex symmetric form[☆]

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Abstract

In this article a new unitary similarity transformation of a normal matrix to complex symmetric form will be discussed. A constructive proof as well as some properties and examples will be given.

Keywords: Normal, complex symmetric, unitary similarity transformation

1. Introduction

Algorithms for computing eigenvalues and singular values of matrices are amongst the most important ones in numerical linear algebra. An incredible range of various methods exist (1; 2), iterative (e.g. Lanczos; Arnoldi) as well as the so called direct methods (e.g. divide-and-conquer algorithms; *GR*-methods). Many of the procedures for computing eigenvalues and/or singular values are based on the *QR*-method.

For a matrix A the *QR*-method consists of two steps. A preprocessing step to transform the matrix A to a suitable shape admitting low cost iterations in the second step. The second step consists of repeatedly applying *QR*-steps on the matrix until the eigenvalues are revealed.

The first step is essential since generically it reduces the global computational complexity of the next step with one order (e.g. from $O(n^4)$ to $O(n^3)$ for an arbitrary matrix A). The definition of suitable shape depends heavily on the matrix type used. For example for (skew)-Hermitian and unitary matrices, which are also normal, but also for more exotic classes such as Hamiltonian, symplectic, product matrices and so forth, different types of suitable shapes exist (2).

Only for specific subclasses of the normal matrix class an efficient preprocessing step is developed, resulting in an $O(n)$ parameter representation of the transformed matrix. In general (3–6) it is not possible to obtain e.g. a band form for a normal matrix. In (7; 8) an algorithm is proposed, transforming a normal matrix to a band form with increasing band-width. In case of Hermitian and skew-Hermitian matrices this coincides with the well-known tridiagonalization procedures.

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In this article a constructive procedure will be given to perform a unitary similarity transformation of a normal matrix, having distinct singular values, to complex symmetric form. The presented algorithm is capable of performing the transformation in a finite number of floating point operations. In a forthcoming article we will discuss the possibility and present a new method for computing eigenvalues of some normal matrices based on this transformation. Here, the reduction as well as some of its properties will be studied. Another solution to the problem was proposed in (9). The proposed algorithm is essentially different, since it is based on the Toeplitz decomposition of the normal matrix, whereas the approach presented here acts directly on the normal matrix.

The article is organized as follows. Section 2 introduces some basic terminology and preliminary results. In Section 3 the main theorem is formulated and proven. Section 4 discusses some extra properties and gives some examples. Some comments related to the case in which the singular values are not distinct are given in Section 5. We end with the conclusions section.

2. Preliminary results

The following notation is used throughout the manuscript: A^T denotes the transpose of A ; \bar{A} the element wise conjugate and $A^H = \bar{A}^T$ is the complex or Hermitian conjugate of the matrix A . The square root of one: $\iota = \sqrt{-1}$.

It is well-known that any normal matrix A ($AA^H = A^HA$) has a full set of orthogonal eigenvectors: $A = Q\Delta Q^H$, with Q unitary, and Δ a diagonal matrix containing the eigenvalues, which is called the eigenvalue decomposition. Hermitian ($A^H = A$), skew-Hermitian ($A^H = -A$) and unitary matrices ($A^HA = AA^H = I$) are special types of normal matrices having respectively a real and purely imaginary spectrum and a spectrum lying on the unit circle.

The singular value decomposition $A = U\Sigma V^H$, with U and V unitary and Σ a diagonal matrix containing the singular values, of a normal matrix is closely related to its eigenvalue decomposition. The following relations with the eigenvalue decomposition hold $\Delta = \Sigma D$ (or $|\Delta| = \Sigma$), with D a unitary diagonal matrix $D = V^H U$.

A matrix $A \in \mathbb{C}^{n \times n}$ is said to be complex symmetric if $A = A^T$. Complex symmetric matrices admit a symmetric singular value decomposition or Takagi factorization (10; 11): $A = Q\Sigma Q^T$, having Q unitary and Σ a diagonal matrix with the singular values of A on its diagonal.

Important to know is that based on the symmetric Jordan canonical form one can easily prove that every matrix A is similar to a complex symmetric matrix (11, Theorem 4.4.9) and (12). For a normal matrix the eigenvalue decomposition also reveals a unitary similarity transform to complex symmetric form. In this article, however, we will prove the existence of a finite (non-iterative) method for executing this transformation in case the normal matrix has distinct singular values.

3. Unitary similarity to complex symmetric form

The next theorem provides in fact a finite constructive method for transforming a normal matrix A to normal complex symmetric form based on e.g. Householders or Givens transformations.

Theorem 1. *Suppose A to be normal, having distinct singular values and $A = UBV^H$, with U, V unitary and B a real upper bidiagonal matrix. Then the unitary similarity transformation $U^H A U$ results in a normal complex symmetric matrix.*

Proof. Assume the factorization $B = U^H AV$, satisfying the constraints above, is known (see e.g. (1; 13; 14) for algorithms computing this factorization). The unitary matrices U and V can either be combinations of Householder and/or Givens transformations, but are constructed such that the matrix B is real.

We will prove now that the unitary similar matrix $A_U = U^H AU$ is normal complex symmetric. Only the complex symmetry $A_U^T = A_U$ needs a formal proof, since normality is preserved under similarity. The following relations hold for $A_U A_U^H$:

$$A_U A_U^H = (U^H AU)(U^H A^H U) = U^H A A^H U = (U^H AV)(V^H A^H U) = B B^H. \quad (1)$$

The matrix B is real bidiagonal implying that $B B^H$ is real, symmetric and tridiagonal. Hence we have $B B^H = \overline{B B^H}$. This gives us for Equation (1):

$$A_U A_U^H = B B^H = \overline{B B^H} = \overline{A_U A_U^H}^T = \overline{A_U}^T A_U^T \quad (2)$$

Assume we have the following eigenvalue decomposition¹ for $A_U = Q \Delta Q^H$ (see Section 2). Equation (2) leads us to the following two eigenvalue decompositions for the matrix product $A_U A_U^H$:

$$Q |\Delta|^2 Q^H = A_U A_U^H = \overline{A_U}^T A_U^T = \overline{Q} |\Delta|^2 Q^T. \quad (3)$$

Since all singular values of A_U are distinct (remember that for normal matrices $|\Delta| = \Sigma$), the eigenvalue decompositions in Equation (3) are essentially unique² and we obtain $Q D = \overline{Q}$, with D a unitary diagonal matrix. This implies $A_U = Q \Delta Q^H = Q \Delta \overline{Q}^T = Q \Delta (Q D)^T = (Q D^{1/2}) \Delta (Q D^{1/2})^T$, indicating that $A_U^T = A_U$ and hence the matrix A_U is complex symmetric. \square

Remark 1. In the proof above we assumed the matrix B to be real. When constructing the matrices U and V based on Householder transformations as in the Golub-Kahan bidiagonalization procedure (15) one can easily impose all diagonal and subdiagonal elements to be real, except for the last one b_{nn} . A left multiplication with a unitary diagonal matrix can rotate b_{nn} such that it becomes real and hence we have a constructive procedure for obtaining the unitary matrix U .

When redoing the proof above by using the transformation $A_V = V^H AV$, we can see that all statements remain valid and hence the matrix A_V will also be normal complex symmetric. Remark that in this case it is even not required that b_{nn} is real (Equation (2) needs to be altered to obtain $B^H B = \overline{B^H B}$, which holds even for complex b_{nn}).

4. Extra properties

The constraint of B being real in Theorem 1 can be omitted. In this case we do not get a complex symmetric matrix anymore, but a matrix self-adjoint for a specific unitary diagonal weight matrix.

In this section results and notations are used from (16; 17). Let us define the bilinear form $\langle \cdot, \cdot \rangle_\Omega$ as $\langle \mathbf{x}, \mathbf{y} \rangle_\Omega = \mathbf{x}^T \Omega \mathbf{y}$. The bilinear form is assumed to be non-degenerate, meaning that Ω is

¹Note that even in case A is real the matrices Q and Δ can be complex.

²Essentially unique means identical up to a unitary diagonal scaling.

nonsingular. For diagonal Ω the bilinear form is referred to as a scalar product with weight matrix Ω . The adjoint of a matrix A with regard to $\langle \cdot, \cdot \rangle_\Omega$ is the matrix A^\star such that $\langle A\mathbf{x}, \mathbf{y} \rangle_\Omega = \langle \mathbf{x}, A^\star \mathbf{y} \rangle_\Omega$, for $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ with \mathbb{F} either \mathbb{C} or \mathbb{R} . The adjoint admits a closed formula:

$$A^\star = \Omega^{-1} A^T \Omega. \quad (4)$$

The matrix A^\star is called the adjoint of A with regard to the weight matrix Ω . A matrix A is self-adjoint means that $A^\star = A$.

Theorem 2. *Let A be normal, having distinct singular values and $A = UBV^H$, with U, V unitary and B a bidiagonal matrix. We have that $U^H A U$ and $V^H A V$ will be self-adjoint with regard to a unitary diagonal matrix Ω_U and Ω_V respectively.*

Proof. Construct two unitary diagonal matrices D_U and D_V , such that the new bidiagonal matrix \hat{B} with $D_U^H (U^H A V) D_V = D_U^H B D_V = \hat{B}$, is real. Let $A_U = U^H A U$, Theorem 1 states that $D_U^H A_U D_U$ is complex symmetric. This together with the unitarity of D_U implies $(D_U^{-1})^T = \overline{D_U}$

$$D_U^{-1} A_U D_U = D_U^H A_U D_U = (D_U^H A_U D_U)^T = D_U A_U^T D_U^{-1}. \quad (5)$$

It remains to prove that there exists a Ω_U such that A_U is self-adjoint with regard to this matrix Ω_U . Take $\Omega_U = D_U^{-2}$, which is unitary diagonal. We obtain (use Equation (4) and (5)) $A_U^\star = \Omega_U^{-1} A_U^T \Omega_U = D_U^2 A_U^T D_U^{-2} = A_U$, proving thereby the self-adjointness. The proof for A_V proceeds identical. \square

Remark 2. *In practice Theorem 2 means for $A_U = (a_{ij})_{ij}$ that $|a_{ij}| = |a_{ji}|$, for all feasible i, j .*

Corollary 1. *Suppose A to be normal, having distinct singular values and $A = UBV^H$, with U, V unitary and B an upper bidiagonal matrix. Then there exist unitary diagonal matrices D_U and D_V such that $D_U^H (U^H A U) D_U$ and $D_V^H (V^H A V) D_V$ are complex symmetric matrices.*

This means that in fact applying a similarity transform with U or V on A always results in a matrix which is unitary scalable to complex symmetric form.

Corollary 2. *Under the conditions of Theorem 1 one can obtain A_U and A_V of complex symmetric form and hence self-adjoint for the standard scalar product. This means that the weight matrix Ω equals the identity.*

In Table 1 the outcome of applying the unitary similarity transform on several classes of matrices is shown. We explicitly assume that the matrices considered have all singular values distinct, excluding thereby some important cases such as orthogonal and unitary matrices. Two transformations are considered: an arbitrary transformation, this means the transformation from Corollary 1 in which A_U is not necessarily complex symmetric and the unitary similarity transform to complex symmetric form from Theorem 1. With the field $i\mathbb{R}$ we denote purely imaginary numbers. We remark that the initial structure never gets lost. This means that when transforming a skew-Hermitian matrix to complex symmetric form, the resulting matrix will be both skew-Hermitian and complex symmetric. When performing a unitary similarity transformation on a real matrix, we silently assume the unitary matrices to be orthogonal which means real unitary.

Most of the relations admit an easy proof (see Examples 1 and 2). The skew-symmetric case however shows a strange outcome. The resulting matrix is not complex symmetric but

Table 1: Outcome of some unitary similarity transformations applied on specific normal matrices. All matrices are assumed to have distinct singular values except in the skew-symmetric case pairs of equal singular values are allowed.

		Reduction Types (Relations for a_{ij} , with $A_U = (a_{ij})_{ij}$.)			
Matrix Type	\mathbb{F}	Arb. ($ a_{ij} = a_{ji} $)		Cplx. Sym. ($a_{ij} = a_{ji}$)	
Normal	\mathbb{R}	Sym.	\mathbb{R}	Sym.	\mathbb{R}
Sym.	\mathbb{R}	Sym.	\mathbb{R}	Sym.	\mathbb{R}
Skew-Sym.	\mathbb{R}	Skew-Sym Block Anti-Diag.	\mathbb{R}	Skew-Sym Block Anti-Diag.	\mathbb{R}
Normal	\mathbb{C}	Normal	\mathbb{C}	Cplx. Sym.	\mathbb{C}
Herm.	\mathbb{C}	Herm.	\mathbb{C}	Sym.	\mathbb{R}
Skew-Herm.	\mathbb{C}	Skew-Herm.	\mathbb{C}	Cplx. Sym.	$i\mathbb{R}$

skew-symmetric and of block anti-diagonal form³. Theorem 1 is, however, not applicable to this matrix since all its eigenvalues appear in conjugate pairs and therefore this matrix does not have distinct singular values. We come back to this in Section 5.

Example 1. Applying the complex symmetric similarity transform on a Hermitian matrix we have $A_U^T = A_U = A_U^H = \bar{A}_U^T$, hence $A_U = \bar{A}_U$ and A_U becomes real.

Example 2. Applying the complex symmetric similarity transform on a skew-Hermitian matrix we have $A_U^T = A_U = -\bar{A}_U^T$, implying $\bar{A}_U = -A_U$ and therefore the matrix has all elements in $i\mathbb{R}$.

5. Normal matrices with non-distinct singular values

When a normal matrix has non-distinct singular values, such as for example a unitary matrix, having all singular values equal to 1, the statements of Theorem 1 do not necessarily hold anymore. Let us reconsider the proof. Everything up to and including Equation (2) still holds. In fact even Equation (3) still holds but we cannot conclude anymore that Q and \bar{Q} are essentially unique. So in case of a unitary matrix having all singular values equal to 1 there is not much one can say about the relation between Q and \bar{Q} .

Let us reconsider, however, the skew-symmetric case from Section 4 (even matrix size). The matrix A_U (with A_U as in Theorem 1) is skew-symmetric and hence admits a factorization (11, Corollary 2.5.14) QJQ^T , where Q is orthogonal. The matrix J is anti-block diagonal and the upper right block $J_{12} = \text{diag}(\beta_1, \dots, \beta_{n/2})$, where $\pm i\beta_i$ are the eigenvalues of A_U . The lower left block $J_{21} = -J_{12}$. The eigenvectors of A_U are given by $\mathbf{q}_i \pm i\mathbf{q}_{n/2+i}$. This factorization is essentially unique, when all β_i are different.

Consider now the matrix product $A_U A_U^H = A_U A_U^T = QJQ^T Q(-J)Q^T = Q(-J^2)Q^T$, where $-J^2$ contains the eigenvalues of $A_U A_U^H$ giving us thus an eigenvalue decomposition of $A_U A_U^H$. The matrix $A_U A_U^H$ is also tridiagonal by Equation (1). An irreducible tridiagonal matrix has all eigenvalues different from each other. Since $A_U A_U^H$ has, however, pairs of equal eigenvalues the matrix must be block diagonal and admits therefore another eigenvalue decomposition of the form $A_U A_U^H = \hat{Q}(-J^2)\hat{Q}^T$, with \hat{Q} orthogonal and block diagonal.

³This means block diagonal along the anti-diagonal.

Properties of eigenvalue decompositions imply that the eigenvectors of Q must be linear combinations of the eigenvectors of \hat{Q} . Due to the block diagonal structure of \hat{Q} , there exist diagonal matrices D_{ij} such that $Q = \begin{bmatrix} \hat{Q}_{11}D_{11} & \hat{Q}_{11}D_{12} \\ \hat{Q}_{22}D_{21} & \hat{Q}_{22}D_{22} \end{bmatrix}$.

Reconsidering the product $A_U = QJQ^T$, we get that $(A_U)_{11} = \hat{Q}_{11}D_{12}(-J_{12})D_{11}Q_{11}^T + \hat{Q}_{11}D_{11}J_{12}D_{12}\hat{Q}_{11}^T = 0$. Similarly one can deduce that $(A_U)_{22}$ is zero. Hence, the matrix A_U is block anti-diagonal as indicated in Table 1.

Based on the singular value decomposition of A_U , one can prove that Q has $Q_{12} = Q_{21} = 0$. This implies that the eigenvectors of A_U have the following special structure⁴ $\mathbf{q}_i \pm i\mathbf{q}_{n/2+i} = [\mathbf{q}_{1i}, \dots, \mathbf{q}_{n/2,i}, \pm i\mathbf{q}_{n/2+1,n/2+i}, \dots, \pm i\mathbf{q}_{n/2+1,n}]$. This means that the first $n/2$ elements of the eigenvectors are real and the last $n/2$ elements are always purely imaginary.

6. Conclusions

In this article a new finite constructive procedure was described for transforming a normal matrix, having distinct singular values, via unitary similarity transformations to complex symmetric form. Some properties were deduced and some examples of the outcome when applied to specific classes such as Hermitian, skew-Hermitian were given.

A test version of the software can be downloaded from the author's homepage.

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⁴Eigenvectors are not unique, but a unitary diagonal scaling of the eigenvectors exists such that the following holds.